

A counter-example to the Andreotti-Grauert conjecture

Youssef ALAOUI

Summary. Let X be an analytic complex space which is q -complete. Then it follows from a theorem of Andreotti-Grauert [1] that $H^p(X, \mathcal{F}) = 0$ for every coherent analytic sheaf \mathcal{F} on X if $p \geq q$. Until now it is not known if these two conditions are equivalent. The aim of this article is to give a counterexample to the converse of this statement. We show that there exist for each $n \geq 3$ open sets $\Omega \subset \mathbb{C}^n$ such that $H^{n-1}(\Omega, \mathcal{F}) = 0$ for every $\mathcal{F} \in \text{coh}(\Omega)$ but Ω is not $(n-1)$ -complete.

Key words: Stein spaces; q -convex functions; q -complete and q -Runge complex spaces.

2000 MS Classification numbers: 32E10, 32E40.

1 Introduction

In 1962, Andreotti and Grauert [1] showed finiteness and vanishing theorems for cohomology groups of analytic spaces under geometric conditions of q -convexity. Since then the question whether the reciprocal statements of these theorems are true have been subject to extensive studies, where for $q > 1$ more specific assumptions have been added. For example, it is known from the theory of Andreotti-Grauert [1] that a q -complete complex space is always cohomologically q -complete, but it is not known if these two conditions are equivalent except when X is a Stein manifold, $\Omega \subset X$ is cohomologically q -complete with respect to O_Ω and Ω has a smooth boundary [5].

The aim of the present article is to give a counterexample to the conjecture posed by Andreotti and Grauert [1] to show that a cohomologically q -complete space is not necessarily q -complete.

More precisely we will show

theorem 1 -For each integer $n \geq 3$, there is a domain $\Omega \subset \mathbb{C}^n$ which is cohomologically $(n-1)$ -complete but Ω is not $(n-1)$ -complete.

2 Preliminaries

Let ϕ be a real valued function in $C^\infty(\Omega)$, where Ω is an open set in \mathbb{C}^n with complex coordinates z_1, \dots, z_n . Then we say that ϕ is q -convex if its complex Hessian $(\frac{\partial^2 \phi(z)}{\partial z_i \partial \bar{z}_j})_{1 \leq i, j \leq n}$ has at most $q-1$ negative or zero eigenvalues for every $z \in \Omega$.

A function $\rho \in C^o(\Omega, \mathbb{R})$ is said to be q -convex with corners, if every point of Ω admits a neighborhood U on which there exist finitely many q -convex functions ϕ_1, \dots, ϕ_l such that $\rho|_U = \max(\phi_1, \dots, \phi_l)$.

The open set Ω is called q -complete if there exists a smooth q -convex exhaustion function on Ω .

We say that Ω is cohomologically q -complete, if for every coherent analytic sheaf \mathcal{F} on Ω , the cohomology group $H^p(\Omega, \mathcal{F}) = 0$ for all $p \geq q$.

Finally, an open subset D of Ω is called q -Runge, if for each compact set $K \subset \Omega$, there is a q -convex exhaustion function $\phi \in C^\infty(\Omega)$ such that

$$K \subset \{z \in \Omega : \phi(z) < c\} \subset\subset D$$

It is known from [1] that if D is q -Runge in Ω , then for every coherent analytic sheaf $\mathcal{F} \in \text{coh}(\Omega)$, the restriction map $H^p(\Omega, \mathcal{F}) \longrightarrow H^p(D, \mathcal{F})$ has dense image for all $p \geq q-1$, or equivalently, for every open covering $\mathcal{U} = (U_i)_{i \in I}$ of Ω with a fundamental system of Stein neighborhoods of Ω , the restriction map between spaces of cocycles

$$Z^p(\mathcal{U}, \mathcal{F}) \longrightarrow Z^p(\mathcal{U}|_D, \mathcal{F})$$

has dense range for $p \geq q-1$.

3 Proof of theorem 1

We consider for $n \geq 3$ the functions $\phi_1, \phi_2 : \mathbb{C}^n \rightarrow \mathbb{R}$ defined by

$$\begin{aligned}\phi_1(z) &= \sigma_1(z) + \sigma_1(z)^2 + N||z||^4 - \frac{1}{4}||z||^2, \\ \phi_2(z) &= -\sigma_1(z) + \sigma_1(z)^2 + N||z||^4 - \frac{1}{4}||z||^2,\end{aligned}$$

where $\sigma_1(z) = \text{Im}(z_1) + \left(\sum_{i=3}^n |z_i|^2\right) - |z_2|^2$, $z = (z_1, z_2, \dots, z_n)$, and $N > 0$ a positive constant. Then, if N is large enough, the functions ϕ_1 and ϕ_2 are $(n-1)$ -convex on \mathbb{C}^n and, if $\rho = \text{Max}(\phi_1, \phi_2)$, then, for $\varepsilon_o > 0$ small enough, the set $D_{\varepsilon_o} = \{z \in \mathbb{C}^n : \rho(z) < -\varepsilon_o\}$ is relatively compact in the unit ball $B = B(0, 1)$, if N is sufficiently large. This is a special case of an example given and utilized by Diederich and Fornaess in a different context. (See [4]).

Proposition 1

In the situation described above for every coherent analytic sheaf \mathcal{F} on D_{ε_o} , the cohomology groups $H^p(D_{\varepsilon_o}, \mathcal{F})$ vanish for all $p \geq n-1$.

Proof.

We consider the set A of all real numbers $\varepsilon \geq \varepsilon_o$ such that $H^{n-1}(D_\varepsilon, \mathcal{F}) = 0$, where $D_\varepsilon = \{z \in \mathbb{C}^n : \rho(z) < -\varepsilon\}$. To prove proposition 1, it will be sufficient to show that

- (a) $A \neq \emptyset$ and, if $\varepsilon \in A$ and $\varepsilon' > \varepsilon$, then $\varepsilon' \in A$.
- (b) if $\varepsilon_j \searrow \varepsilon$ and $\varepsilon_j \in A$ for all j , then $\varepsilon \in A$.
- (c) if $\varepsilon \in A$, $\varepsilon > \varepsilon_o$, there exists $\varepsilon_o \leq \varepsilon' < \varepsilon$ such that $\varepsilon' \in A$.

We first prove (a). Choose $\varepsilon_1 > \varepsilon_o$ such that

$$-\varepsilon_1 < \inf_{z \in \partial D_{\varepsilon_o}} \{\phi_i(z), i = 1, 2\},$$

and set $D_i = \{z \in D_{\varepsilon_o} : \phi_i(z) < -\varepsilon\}$ for $\varepsilon \geq \varepsilon_1$. Then, if D_ε is not empty, the sets $D_i \subset\subset D_{\varepsilon_o}$ are clearly $(n-1)$ -complete, since $\frac{-1}{\phi_i + \varepsilon}$ is a $(n-1)$ -convex exhaustion function on D_i . Therefore, using the exact sequence of cohomology associated to the Mayer-Vietoris sequence

$$\rightarrow H^{n-1}(D_1, \mathcal{F}) \oplus H^{n-1}(D_2, \mathcal{F}) \rightarrow H^{n-1}(D_\varepsilon, \mathcal{F}) \rightarrow H^n(D_1 \cup D_2, \mathcal{F}) \rightarrow$$

one obtains $H^{n-1}(D_\varepsilon, \mathcal{F}) = 0$ and, obviously $[\varepsilon_1, +\infty[\subset A$.

Let now $\varepsilon \in A$ and $\varepsilon' > \varepsilon$. Then $D_{\varepsilon'} \subset\subset D_\varepsilon$ is n -Runge in B . Indeed, if $K \subset D_{\varepsilon'}$ is a compact set, there exists a $(n-1)$ -convex exhaustion function $\psi_i \in C^\infty(B)$, such that $K \subset \{\psi_i < 0\} \subset\subset D_i = \{z \in B : \phi_i(z) < -\varepsilon'\}$, $i = 1, 2$ because D_i is obviously $(n-1)$ -Runge in B , the function ϕ_i being $(n-1)$ -convex and B is Stein. Then a suitable smooth n -convex approximation of $\max(\psi_1, \psi_2)$ ([4]) shows that $D_{\varepsilon'}$ is n -Runge in B . It follows from [3] that $D_\varepsilon \setminus D_{\varepsilon'}$ has no compact connected components and, therefore the restriction map

$$H^{n-1}(D_\varepsilon, \mathcal{F}) \longrightarrow H^{n-1}(D_{\varepsilon'}, \mathcal{F})$$

has dense image. This proves that $H^{n-1}(D_{\varepsilon'}, \mathcal{F}) = 0$ and $\varepsilon' \in A$.

The proof of statement (c) will result from two lemmas. First note that if $\varepsilon > \varepsilon_o$, then $\dim_{\mathbb{C}} H^{n-1}(D_\varepsilon, \mathcal{F}) < \infty$. In fact, choose finitely many Stein open sets $U_i \subset\subset D_{\varepsilon_o}$, $i = 1, \dots, k$, such that $\partial D_\varepsilon \subset \bigcup_{i=1}^k U_i$.

Let $\theta_j \in C_o^\infty(U_j, \mathbb{R}^+)$ such that $\sum_{j=1}^k \theta_j(x) > 0$ at any point $x \in \partial D_\varepsilon$. Let also

$c_i > 0$ be sufficiently small constants such that the functions $\phi_i - \sum_{i=1}^j c_i \theta_i$ are $(n-1)$ -convex for $i = 1, 2$ and $1 \leq j \leq k$. We now define the continuous functions $\rho_j : \mathbb{C}^n \rightarrow \mathbb{R}$ by

$$\rho_j = \rho - \sum_{i=1}^j c_i \theta_i, j = 1, \dots, k$$

Then ρ_j are $(n-1)$ -convex with corners and, if $D_j = \{z \in D_{\varepsilon_o} : \rho_j(z) < -\varepsilon\}$, $j = 1, \dots, k$, and $D_o = D_\varepsilon$, then $D_o \subset D_1 \subset \dots \subset D_k$,

$D_o \subset\subset D_k \subset\subset D_{\varepsilon_o}$ and $D_j \setminus D_{j-1} \subset\subset U_j$ for $j = 1, \dots, k$. Furthermore we remark that $H^{n-1}(D_j \cap U_l, \mathcal{F}) = 0$ for $0 \leq j \leq k$, $1 \leq l \leq k$. In fact, since $D_j \cap U_l$ can be written in the form $D_j \cap U_l = B_{1,j} \cap B_{2,j}$ where

$B_{i,j} = \{z \in U_l : \phi_i - \sum_{r=1}^j c_r \theta_r < -\varepsilon\}$, $i = 1, 2$, is $(n-1)$ -complete because U_l is

Stein and $\phi_i - \sum_{r=1}^j c_r \theta_r$ is $(n-1)$ -convex in U_l , then from the Mayer-Vietoris sequence

$$\rightarrow H^{n-1}(B_{1,j}, \mathcal{F}) \oplus H^{n-1}(B_{2,j}, \mathcal{F}) \rightarrow H^{n-1}(D_j \cap U_l, \mathcal{F}) \rightarrow H^n(B_{1,j} \cup B_{2,j}, \mathcal{F}) \rightarrow$$

it follows that $H^{n-1}(D_j \cap U_l, \mathcal{F}) = 0$. Now using the Mayer-Vietoris sequence

$$\rightarrow H^{n-1}(D_j, \mathcal{F}) \rightarrow H^{n-1}(D_{j-1}, \mathcal{F}) \oplus H^{n-1}(D_j \cap U_j, \mathcal{F}) \rightarrow H^{n-1}(D_{j-1} \cap U_j, \mathcal{F}) \rightarrow$$

and noting that $H^{n-1}(D_j \cap U_j, \mathcal{F}) = H^{n-1}(D_{j-1} \cap U_j, \mathcal{F}) = 0$ for all $1 \leq j \leq k$, we find that $H^{n-1}(D_k, \mathcal{F}) \rightarrow H^{n-1}(D_\varepsilon, \mathcal{F})$ is surjective. This implies that $\dim_{\mathbb{C}} H^{n-1}(D_\varepsilon, \mathcal{F}) < \infty$. (Cf. Proof of theorem 11 of [1]).

We now put $\psi_{i,j} = \phi_i - \sum_{i=1}^j c_i \theta_i$, $\psi_{i,o} = \phi_i$, $i = 1, 2$, $1 \leq j \leq k$ and define the open sets $D_{j,i}$, as follows $D_{1,o} = \{\phi_1 < -\varepsilon, \phi_2 < -\varepsilon\}$, $D_{1,1} = \{\psi_{1,1} < -\varepsilon, \phi_2 < -\varepsilon\}$, $D_{1,2} = \{\psi_{1,1} < -\varepsilon, \psi_{2,1} < -\varepsilon\}$. And for $2 \leq j \leq k$ we set $D_{j,o} = \{\psi_{1,j-1} < -\varepsilon, \psi_{2,j-1} < -\varepsilon\}$, $D_{j,1} = \{\psi_{1,j} < -\varepsilon, \psi_{2,j-1} < -\varepsilon\}$, $D_{j,2} = \{\psi_{1,j} < -\varepsilon, \psi_{2,j} < -\varepsilon\}$. Obviously, $D_{1,o} = D_o$ and $D_{j,2} = D_{j+1,o} = D_j$ for $1 \leq j \leq k$. Therefore $D_o = D_{1,o} \subset D_{1,1} \subset D_{1,2} = D_{2,o} \subset \dots \subset D_{k,o} \subset D_{k,1} \subset D_{k,2} = D_k$

lemma 1 - *The restriction map $H^{n-2}(D_{j+1}, \mathcal{F}) \rightarrow H^{n-2}(D_j, \mathcal{F})$ has a dense image for all $0 \leq j \leq k-1$*

Proof

We first prove that for all $1 \leq j \leq k$ and $0 \leq i \leq 2$ the homology groups $H_p(D_{j,i}, \mathbb{C})$ vanish for every $p \geq 2n-2$. For this, we define $A_{j,1} = \{z \in B : \psi_{1,j} < -\varepsilon\}$, $A_{j,2} = \{z \in B : \psi_{2,j-1} < -\varepsilon\}$. Then $A_{j,1}$ and $A_{j,2}$ are $(n-1)$ -complete and $(n-1)$ -Runge in B because B is Stein and $\psi_{1,j}$ and $\psi_{2,j-1}$ are $(n-1)$ -convex in B . In particular $H_p(A_{j,i}, \mathbb{C}) = H_p(B, A_{j,i}, \mathbb{C}) = 0$ for $p \geq 2n-1$ and $i = 1, 2$. (See [8]). Since $D_{j,1} = A_{j,1} \cap A_{j,2}$ and $A_{j,1} \cup A_{j,2}$ has no compact connected components, then the Mayer-Vietoris sequence for homology

$$\rightarrow H_{p+1}(A_{j,1} \cup A_{j,2}, \mathbb{C}) \rightarrow H_p(D_{j,1}, \mathbb{C}) \rightarrow H_p(A_{j,1}, \mathbb{C}) \oplus H_p(A_{j,2}, \mathbb{C}) \rightarrow$$

shows that $H_p(D_{j,1}, \mathbb{C}) = 0$ for every $p \geq 2n-1$. Moreover, since B is Stein, it follows from the sequence of homology

$$\rightarrow H_{p+1}(B, A_{j,i}, \mathbb{C}) \rightarrow H_p(A_{j,i}, \mathbb{C}) \rightarrow H_p(B, \mathbb{C}) \rightarrow$$

that $H_p(A_{j,i}, \mathbb{C}) = 0$ for $p \geq 2n - 2$.

Now, since $B \setminus (A_{j,1} \cup A_{j,2})$ has no compact connected components, the $A_{j,i}$ being $(n - 1)$ -Runge in B , it follows from [3] that the natural map

$$H_{2n-1}(A_{j,1} \cup A_{j,2}, \mathbb{C}) \longrightarrow H_{2n-1}(B, \mathbb{C})$$

is injective, which shows $H_{2n-1}(A_{j,1} \cup A_{j,2}, \mathbb{C}) = 0$. Also by the sequence of homology given by the Mayer-Vietoris sequence

$$\rightarrow H_{2n-1}(A_{j,1} \cup A_{j,2}, \mathbb{C}) \rightarrow H_{2n-2}(D_{j,1}, \mathbb{C}) \rightarrow H_{2n-2}(A_{j,1}, \mathbb{C}) \oplus H_{2n-2}(A_{j,2}, \mathbb{C}) \rightarrow$$

we deduce that $H_{2n-2}(D_{j,1}, \mathbb{C}) = 0$. Similarly $H_p(D_{j,i}, \mathbb{C}) = 0$ for every $p \geq 2n - 2$ and all $1 \leq j \leq k$ and $i = 0, 1, 2$. It is of course also clear now that $H_p(D_{j,i} \cap U_l, \mathbb{C}) = 0$ for $p \geq 2n - 2$, $1 \leq j, l \leq k$ and $i = 0, 1, 2$.

Let us now choose $\varepsilon' > \varepsilon_o$ such that $D_k \subset\subset D_{\varepsilon'}$. Then $D_{\varepsilon'} \setminus D_{j,i}$ has no compact connected components for every $1 \leq j \leq k$ and $0 \leq i \leq 2$, since $D_{j,i}$ is obviously n -Runge in B . it follows from [3] that the restriction map $H^{n-1}(D_{\varepsilon'}, \mathcal{F}) \rightarrow H^{n-1}(D_{j,i}, \mathcal{F})$ has a dense image, and therefore $\dim_{\mathbb{C}} H^{n-1}(D_{j,i}, \mathcal{F}) < \infty$ for $1 \leq j \leq k$. and $i = 0, 1, 2$.

It is clear that it is sufficient for the proof of lemma 1 to show that the restriction map

$$H^{n-2}(D_{j+1,1}, \mathcal{F}) \longrightarrow H^{n-2}(D_j, \mathcal{F})$$

has a dense image. We have $D_j \cap U_{j+1} = B_{1,j} \cap B_{2,j}$ and $D_{j+1,1} \cap U_{j+1} = B_{1,j+1} \cap B_{2,j}$ where $B_{i,j} = \{z \in U_{j+1} : \psi_{i,j} < -\varepsilon\}$, $i = 1, 2$, and $B_{1,j+1} = \{z \in U_{j+1} : \psi_{1,j+1}(z) < -\varepsilon\}$. Note also that $B_{i,j}$ are $(n - 1)$ -complete and $(n - 1)$ -Runge in the Stein set U_{j+1} . Then $H_p(B_{i,j}, \mathbb{C}) = 0$ for $p \geq 2n - 2$, $i = 1, 2$, $0 \leq j \leq k - 1$. Therefore using the exact sequences of homology

$$\rightarrow H_{2n-1}(B_{1,j}, \mathbb{C}) \oplus H_{2n-1}(B_{2,j}, \mathbb{C}) \rightarrow H_{2n-1}(B_{1,j} \cup B_{j,2}, \mathbb{C}) \rightarrow H_{2n-2}(D_j \cap U_{j+1}, \mathbb{C}) \rightarrow$$

$$\rightarrow H_{2n-1}(B_{1,j+1}, \mathbb{C}) \oplus H_{2n-1}(B_{2,j}, \mathbb{C}) \rightarrow H_{2n-1}(B_{1,j+1} \cup B_{j,2}, \mathbb{C}) \rightarrow H_{2n-2}(D_{j+1} \cap U_{j+1}, \mathbb{C}) \rightarrow$$

we find that $H_{2n-1}(B_{1,j} \cup B_{j,2}, \mathbb{C}) = H_{2n-1}(B_{1,j+1} \cup B_{j,2}, \mathbb{C}) = 0$. By [3] it follows that the restriction map

$$H^{n-1}(B_{1,j+1} \cup B_{2,j}, \mathcal{F}) \longrightarrow H^{n-1}(B_{1,j} \cup B_{2,j}, \mathcal{F})$$

has dense image. On the other hand, the restriction map

$$H^{n-2}(B_{1,j+1}, \mathcal{G}) \longrightarrow H^{n-2}(B_{1,j}, \mathcal{G})$$

has also a dense range for every coherent analytic sheaf \mathcal{G} on U_{j+1} . (See Andreotti-Grauert [1]). Now consider the commutative diagram given by the Mayer-Vietoris sequence for cohomology

$$\begin{array}{ccccccc} \rightarrow & H^{n-2}(B_{1,j+1}, \mathcal{F}) \oplus H^{n-2}(B_{2,j}, \mathcal{F}) & \rightarrow & H^{n-2}(D_{j+1,1} \cap U_{j+1}, \mathcal{F}) & \rightarrow & H^{n-1}(B_{1,j+1} \cup B_{2,j}, \mathcal{F}) & \rightarrow 0 \\ & \downarrow \rho_1 \oplus id & & \rho_2 \downarrow & & \rho_3 \downarrow & \\ \rightarrow & H^{n-2}(B_{1,j}, \mathcal{F}) \oplus H^{n-2}(B_{2,j}, \mathcal{F}) & \rightarrow & H^{n-2}(D_j \cap U_{j+1}, \mathcal{F}) & \xrightarrow{u} & H^{n-1}(B_{1,j} \cup B_{2,j}, \mathcal{F}) & \rightarrow 0 \end{array}$$

Since u is surjective, then u is open by lemma 3.2 of [2] and, since $\rho_1 \oplus id$ and ρ_3 have dense image, it follows that ρ_2 has also a dense image.

Now since $Supp\theta_j \subset U_j$, $j = 1, \dots, k$, then $D_{j,i+1} \setminus D_{j,i} \subset\subset U_j$ and $D_{j,i+1} = D_{j,i} \cup (D_{j,i+1} \cap U_j)$. So the Mayer-Vietoris sequence for cohomology gives the exactness of the sequence

$$\dots \rightarrow H^{n-2}(D_{j+1,1}, \mathcal{F}) \rightarrow H^{n-2}(D_j, \mathcal{F}) \oplus H^{n-2}(D_{j+1,1} \cap U_{j+1}, \mathcal{F}) \rightarrow H^{n-2}(D_j \cap U_{j+1}, \mathcal{F}) \rightarrow H^{n-1}(D_{j+1,1}, \mathcal{F}) \rightarrow \dots$$

Since $H^{n-2}(D_{j+1,1} \cap U_{j+1}, \mathcal{F}) \rightarrow H^{n-2}(D_j \cap U_{j+1}, \mathcal{F})$ has a dense image and $\dim_{\mathbf{C}} H^{n-1}(D_{j+1,1}, \mathcal{F}) < \infty$, then

$$H^{n-2}(D_{j+1,1}, \mathcal{F}) \rightarrow H^{n-2}(D_j, \mathcal{F})$$

has also a dense image. (This follows by the proof of Proposition 19 of [1]).

lemma 2 -Suppose that $\varepsilon \in A$. Then there is $\varepsilon_o \leq \varepsilon' < \varepsilon$ such that $\varepsilon' \in A$

Proof

Let $\mathcal{V} = (V_i)_{i \in \mathbf{N}}$ be an open covering of D_{ε_o} with a fundamental system of Stein neighborhoods of D_{ε_o} such that if $V_{i_o} \cap \dots \cap V_{i_r} \neq \emptyset$, then $V_{i_o} \cup \dots \cup V_{i_r} \subset D_j$ or $V_{i_o} \cup \dots \cup V_{i_r} \subset U_{j+1} \cap D_{j+1}$.

We first show that $H^{n-1}(D_k, \mathcal{F}) = 0$. We shall prove it assuming that it has already been proved for $j < k$. For this, we consider the Mayer-Vietoris sequence for cohomology

$$\rightarrow H^{n-2}(D_j, \mathcal{F}) \oplus H^{n-2}(D_{j+1} \cap U_{j+1}, \mathcal{F}) \xrightarrow{r^*} H^{n-2}(D_j \cap U_{j+1}, \mathcal{F}) \xrightarrow{j^*} H^{n-1}(D_{j+1}, \mathcal{F}) \xrightarrow{\rho^*}$$

Let ξ be a cocycle in $Z^{n-1}(\mathcal{V}|_{D_{j+1}}, \mathcal{F})$ and let $\rho(\xi)$ be its restriction to a cocycle in $Z^{n-1}(\mathcal{V}|_{D_j}, \mathcal{F})$. Since $\rho(\xi)$ is a coboundary by induction and

$H^{n-1}(D_{j+1} \cap U_{j+1}, \mathcal{F}) = 0$, from the Mayer-Vietoris sequence, it follows that there exist

$$\eta \in Z^{n-2}(\mathcal{V}|_{D_j \cap U_{j+1}}, \mathcal{F}) \text{ and } \mu \in C^{n-2}(\mathcal{V}|_{D_{j+1}}, \mathcal{F})$$

such that $\xi = j(\eta) + \delta\mu$. There exists a sequence $\{\eta_n\} \subset Z^{n-2}(\mathcal{V}|_{D_{j+1} \cap U_{j+1}}, \mathcal{F})$ with $r(\eta_n) - \eta \rightarrow 0$, when $n \rightarrow \infty$. This is possible because $Z^{n-2}(\mathcal{V}|_{D_{j+1} \cap U_{j+1}}, \mathcal{F}) \rightarrow Z^{n-2}(\mathcal{V}|_{D_j \cap U_{j+1}}, \mathcal{F})$ has a dense range. Now choose a sequence $\{\gamma_n\} \subset C^{n-2}(\mathcal{V}|_{D_{j+1}}, \mathcal{F})$ such that $j(r(\eta_n)) = \delta\gamma_n$. Then

$$\xi - \delta\mu - \delta\gamma_n = j(\eta - r(\eta_n))$$

This proves that $\delta\mu + \delta\gamma_n$ converges to ξ when $n \rightarrow \infty$. Since $\dim_{\mathbf{C}} H^{n-1}(D_{j+1}, \mathcal{F}) < \infty$, then the coboundary space $B^{n-1}(\mathcal{V}|_{D_{j+1}}, \mathcal{F})$ is closed in $Z^{n-1}(\mathcal{V}|_{D_{j+1}}, \mathcal{F})$. Therefore $\xi \in B^{n-1}(\mathcal{V}|_{D_{j+1}}, \mathcal{F})$ and $H^{n-1}(D_j, \mathcal{F}) = 0$ for all $0 \leq j \leq k$. On the other hand, there exists $\varepsilon' > 0$ such that $\varepsilon - \varepsilon' > \varepsilon_o$ and $D_{\varepsilon - \varepsilon'} = \{z \in D_{\varepsilon_o} : \rho(z) < \varepsilon' - \varepsilon\} \subset\subset D_k$. Since $D_k \setminus D_{\varepsilon - \varepsilon'}$ has no compact connected components, then $H^{n-1}(D_k, \mathcal{F}) \rightarrow H^{n-1}(D_{\varepsilon - \varepsilon'}, \mathcal{F})$ has a dense image, which means that $H^{n-1}(D_{\varepsilon' - \varepsilon'}, \mathcal{F}) = 0$. This proves that $\varepsilon - \varepsilon' \in A$.

In order to prove statement (b), it is sufficient to show that if $\varepsilon_j \searrow \varepsilon$ and $\varepsilon_j \in A$ for all j , then

$$H^{n-2}(D_{\varepsilon_{j+1}}, \mathcal{F}) \longrightarrow H^{n-2}(D_{\varepsilon_j}, \mathcal{F})$$

has dense image. (Cf. [1, p.250]). To complete the proof of Proposition 1, it is therefore enough to prove the following lemma.

lemma 3 - *The restriction map $H^{n-2}(D_{\varepsilon_o}, \mathcal{F}) \rightarrow H^{n-2}(D_{\varepsilon}, \mathcal{F})$ has dense image for every real number $\varepsilon \geq \varepsilon_o$*

Proof

We consider the set T of all $\varepsilon \geq \varepsilon_o$ such that $H^{n-2}(D_{\varepsilon}, \mathcal{F}) \rightarrow H^{n-2}(D_{\varepsilon_1}, \mathcal{F})$ has dense image for every real number $\varepsilon_1 > \varepsilon$.

To see that $T \neq \emptyset$, we choose $\varepsilon > \varepsilon_o$ such that

$-\varepsilon < \min_{\overline{B} \setminus D_{\varepsilon_o}} \{\phi_i(z), i = 1, 2\}$, and let $\varepsilon_1 > \varepsilon$. If D_{ε_1} is not empty, $D_i = \{z \in B : \phi_i(z) < -\varepsilon_1\}$ and $D'_i = \{z \in B : \phi_i(z) < -\varepsilon\}$ are relatively compact in D_{ε_o} , $(n-1)$ -complete and $(n-1)$ -Runge in B . Moreover, D_i is $(n-1)$ -Runge in D'_i . In fact, let $K \subset D_i$ be a compact set and $\varepsilon_2 > \varepsilon_1$ such that $\phi_i < -\varepsilon_2$ on K . Then $\frac{-1}{\phi_i + \varepsilon}$ is a $(n-1)$ -convex exhaustion function on D'_i such that

$$K \subset \{z \in D'_i : \frac{-1}{\phi_i + \varepsilon} < \frac{1}{\varepsilon_2 - \varepsilon}\} = \{z \in D'_i : \phi_i(z) < -\varepsilon_2\} \subset\subset D_i$$

Therefore $H^{n-2}(D'_i, \mathcal{F}) \rightarrow H^{n-2}(D_i, \mathcal{F})$ has dense image for $i = 1, 2$.

The same argument used in the proof of lemma 1 shows that

$H_{2n-1}(D_1 \cup D_2, \mathbb{C}) = H_{2n-1}(D'_1 \cup D_2, \mathbb{C}) = H_{2n-1}(D'_1 \cup D'_2, \mathbb{C}) = 0$ and therefore $H^{n-1}(D'_1 \cup D_2, \mathcal{F}) \rightarrow H^{n-1}(D_1 \cup D_2, \mathcal{F})$, and

$H^{n-1}(D'_1 \cup D'_2, \mathcal{F}) \rightarrow H^{n-1}(D'_1 \cup D_2, \mathcal{F})$ have dense image. Consequently we can show exactly as in lemma 1 page 5 that if $D_{\varepsilon,1} = D'_1 \cap D_2$, then we have the density of the image of the restriction maps $H^{n-2}(D_{\varepsilon,1}, \mathcal{F}) \rightarrow H^{n-2}(D_{\varepsilon,1}, \mathcal{F})$, and $H^{n-2}(D_{\varepsilon}, \mathcal{F}) \rightarrow H^{n-2}(D_{\varepsilon,1}, \mathcal{F})$. This proves that $\varepsilon \in T$ and, clearly $[\varepsilon, +\infty[\subset T$.

Let now $\varepsilon_j \in T$, $j \geq 0$, such that $\varepsilon_j \searrow \varepsilon$, and let $\mathcal{U} = (U_i)_{i \in I}$ be a Stein open covering of D_{ε_o} with a countable base of open subsets of D_{ε_o} . Then the restriction map between spaces of cocycles

$Z^{n-2}(\mathcal{U}|_{D_{\varepsilon_{j+1}}}, \mathcal{F}) \rightarrow Z^{n-2}(\mathcal{U}|_{D_{\varepsilon_j}}, \mathcal{F})$ has dense image for $j \geq 0$. Let $\varepsilon' > \varepsilon$ and $j \in \mathbb{N}$ such that $\varepsilon' > \varepsilon_j$. By [1, p.246], the restriction map

$Z^{n-2}(\mathcal{U}|_{D_{\varepsilon}}, \mathcal{F}) \rightarrow Z^{n-2}(\mathcal{U}|_{D_{\varepsilon_j}}, \mathcal{F})$ has dense image. Since $\varepsilon_j \in T$, then $Z^{n-2}(\mathcal{U}|_{D_{\varepsilon_j}}, \mathcal{F}) \rightarrow Z^{n-2}(\mathcal{U}|_{D_{\varepsilon'}}, \mathcal{F})$ has also dense image, and hence $\varepsilon \in T$.

To prove that T is open in $[\varepsilon_o, +\infty[$ it is sufficient to show that if $\varepsilon \in T$, $\varepsilon > \varepsilon_o$, then there is $\varepsilon_o < \varepsilon' < \varepsilon$ such that $\varepsilon' \in T$. But this can be done in the same way as in the proof of lemma 1. We consider a finite covering $(U_i)_{1 \leq i \leq k}$ of ∂D_{ε} by Stein open sets $U_i \subset\subset D_{\varepsilon_o}$ and compactly supported functions $\theta_i \in C_o^\infty(U_i)$, $\theta_j \geq 0$, $j = 1, \dots, k$ such that $\sum_{i=1}^k \theta_i(x) > 0$ at any point of ∂D_{ε} . Define $D_j = \{z \in D_{\varepsilon_o} : \rho_j(z) < -\varepsilon\}$ where

$$\rho_j(z) = \max(\phi_1 - \sum_{i=1}^j c_i \theta_i, \phi_2 - \sum_{i=1}^j c_i \theta_i) \text{ with } c_i > 0 \text{ sufficiently small so that}$$

$$\psi_{i,j} = \phi_i - \sum_{i=1}^j c_i \theta_i \text{ are still } (n-1)\text{-convex for } i = 1, 2 \text{ and } 1 \leq j \leq k.$$

By lemma 1, the restriction map

$$H^{n-2}(D_k, \mathcal{F}) \longrightarrow H^{n-2}(D_\varepsilon, \mathcal{F})$$

has dense image and, there exists $\varepsilon_o < \varepsilon' < \varepsilon$ such that $D_\varepsilon \subset \subset D_{\varepsilon'} \subset \subset D_k$. For an arbitrary real number $\alpha > 0$ we define $D_j(\alpha) = D_j \cap D_\alpha$, where $D_\alpha = \{\rho < -\alpha\}$ and $j = 0, 1, \dots, k$. We claim that for any $\alpha \geq \varepsilon'$ the restriction map

$$H^{n-2}(D_j(\varepsilon'), \mathcal{F}) \rightarrow H^{n-2}(D_j(\alpha), \mathcal{F})$$

has dense image for all $0 \leq j \leq k$.

Note that this is clearly satisfied for $j = 0$, since $D_o(\varepsilon') = D_\varepsilon$ and $\varepsilon \in T$. For $j \geq 1$, we first check that $H_p(D_j(\alpha), \mathbb{C}) = 0$ for $p \geq 2n - 2$. Since clearly $\mathbb{C}^n \setminus (D_\alpha \cup D_j)$ has no compact connected components, it follows from [3] that $H_{2n-1}(D_\alpha \cup D_j, \mathbb{C}) \rightarrow H_{2n-1}(\mathbb{C}^n, \mathbb{C})$ is injective, which means that $H_{2n-1}(D_\alpha \cup D_j, \mathbb{C}) = 0$. Since we have shown in lemma 1 that $H_p(D_\alpha, \mathbb{C}) = H_p(D_j, \mathbb{C}) = 0$ for $p \geq 2n - 2$, it follows from the Mayer-Vietoris sequence

$$\rightarrow H_{p+1}(D_\alpha \cup D_j, \mathbb{C}) \rightarrow H_p(D_j(\alpha), \mathbb{C}) \rightarrow H_p(D_\alpha, \mathbb{C}) \oplus H_p(D_j, \mathbb{C}) \rightarrow$$

that $H_p(D_j(\alpha), \mathbb{C}) = 0$ for $p \geq 2n - 2$. In particular $H_{2n-2}(D_j(\varepsilon'), \mathbb{C}) = H_{2n-2}(D_j(\alpha), \mathbb{C}) = 0$. Also it follows clearly that if $D'_j(\alpha, \varepsilon') = \{\phi_1 < -\varepsilon', \phi_2 < -\alpha, \rho_j < -\varepsilon\} = D_j \cap D(\varepsilon', \alpha)$, where $D(\varepsilon', \alpha) = \{\phi_1 < -\varepsilon', \phi_2 < -\alpha\}$, then $H_p(D'_j(\alpha, \varepsilon'), \mathbb{C}) = H_p(D(\alpha, \varepsilon'), \mathbb{C}) = 0$ for every $p \geq 2n - 2$ and $H_{2n-1}(D_j \cup D_\alpha, \mathbb{C}) = H_{2n-1}(D_j \cup D(\varepsilon', \alpha), \mathbb{C}) = 0$ for all $\alpha \geq \varepsilon'$.

We now just have to repeat again the same argument used in proof of lemma 1 to show that $H^{n-2}(D_j(\varepsilon'), \mathcal{F}) \rightarrow H^{n-2}(D'_j(\alpha, \varepsilon'), \mathcal{F})$ and $H^{n-2}(D'_j(\alpha, \varepsilon'), \mathcal{F}) \rightarrow H^{n-2}(D_j(\alpha), \mathcal{F})$ have dense image. If we take $j = k$, then we find that the restriction map

$$H^{n-2}(D_{\varepsilon'}, \mathcal{F}) \longrightarrow H^{n-2}(D_\alpha, \mathcal{F})$$

has dense image for every $\alpha > \varepsilon'$, which shows that $\varepsilon' \in T$ and $T = [\varepsilon_o, +\infty[$.

End of the proof of theorem 1

Let $0 < \varepsilon < \varepsilon_o$ be such that $D_\varepsilon = \{z \in \mathbb{C}^n : \rho(z) < 0\} \subset \subset B$. Then in view of the proof of Proposition 1 it follows that D_ε is cohomologically $(n-1)$ -complete. We shall prove that D_ε is not $(n-1)$ -complete. In fact, it was shown by Diederich-Fornaess [4] that if $\delta > 0$ is small enough, then the topological sphere $S_\delta = \{z \in \mathbb{C}^n : x_1^2 + |z_2|^2 + \cdots + |z_n|^2 = \delta, y_1 = -\sum_{i=3}^n |z_i|^2 + |z_2|^2\}$ is not homologous to 0 in D_ε . This follows from the fact that the set $E = \{z \in \mathbb{C}^n : x_1 = z_2 = \cdots = z_n = 0\}$ does not intersect D_ε .

We can prove exactly as in lemma 1 that $H_{2n-2}(D_\varepsilon, \mathbb{R}) = 0$. Indeed, We have $D_\varepsilon = D_1 \cap D_2$, where $D_i = \{z \in \mathbb{C}^n : \phi_i(z) < \varepsilon\}$, $i = 1, 2$. Since D_i is $(n-1)$ -Runge in \mathbb{C}^n , then the natural map $H_{2n-2}(D_i, \mathbb{R}) \rightarrow H_{2n-2}(\mathbb{C}^n, \mathbb{R})$ is injective, which implies that $H_{2n-2}(D_i, \mathbb{R}) = 0$ for $i = 1, 2$. We now consider the Mayer-Vietoris sequence for homology

$$\cdots \rightarrow H_{2n-1}(D_1 \cup D_2, \mathbb{R}) \rightarrow H_{2n-2}(D_\varepsilon, \mathbb{R}) \rightarrow H_{2n-2}(D_1, \mathbb{R}) \oplus H_{2n-2}(D_2, \mathbb{R}) = 0$$

Since $\mathbb{C}^n \setminus D_1 \cup D_2$ has no compact irreducible components, then $H_{2n-1}(D_1 \cup D_2, \mathbb{R}) \rightarrow H_{2n-1}(\mathbb{C}^n, \mathbb{R})$ is injective. This proves that $H_{2n-1}(D_1 \cup D_2, \mathbb{R}) = 0$ and, therefore $H_{2n-2}(D_\varepsilon, \mathbb{R}) = 0$.

Suppose now that D_ε is $(n-1)$ -complete. Then there exists a $(n-1)$ -convex exhaustion function $\psi \in C^\infty(D_\varepsilon)$ such that

$$K = \overline{D}_{\varepsilon_o} \subset \tilde{D}_\varepsilon = \{z \in D_\varepsilon : \psi(z) < 0\}$$

Since $S_\delta \subset \tilde{D}_\varepsilon$ and $\tilde{D}_\varepsilon \cap E = \emptyset$, then the sphere S_δ is not homologous to 0 in \tilde{D}_ε . Since, in addition, the levi form of ψ has at least 2 strictly positive eigenvalues, then $H_{2n-2}(\tilde{D}_\varepsilon, \mathbb{R}) \neq 0$. But as \tilde{D}_ε is $(n-1)$ -Runge in D_ε , the natural map $H_{2n-2}(\tilde{D}_\varepsilon, \mathbb{R}) \rightarrow H_{2n-2}(D_\varepsilon, \mathbb{R})$ is injective. Therefore $H_{2n-2}(\tilde{D}_\varepsilon, \mathbb{R}) = 0$, which is a contradiction. This proves that D_ε is not $(n-1)$ -complete.

References

- [1] A. Andreotti and H. Grauert. Théorèmes de finitude pour la cohomologie des espaces complexes. Bull. Soc. Math. France 90 (1962), 153-259.

- [2] M. Coltoiu, On Barth's conjecture concerning $H^{n-1}(\mathbb{P}^n \setminus A, \mathcal{F})$. Nagoya Math. J. Vol. 145 (1997), 99-123.
- [3] M. Coltoiu and A. Silva, Behnke-Stein theorem on complex spaces with singularities. Nagoya Math. J. Vol. 137 (1995), 183-194.
- [4] H. Diederich, J. E. Fornaess, Smoothing q -convex functions and vanishing theorems. Invent. Math. 82. 291-305 (1985)
- [5] M.G. Eastwood and G.V Suria, Cohomologically complete and pseudoconvex domains. Comment. Math. Helv. 55 (1980), 413-426
- [6] G. Lupacchiolu, E.L. Stout, Holomorphic hulls in terms of curves of non-negative divisors and generalized polynomial hulls. manuscripta math. 98, 321-331 (1999)
- [7] M. Peternell, Ein Lefschetz-Satz für Schnitte in projektiv-algebraischen Mannigfaltigkeiten. Math. Ann. 264, 361-388 (1981)
- [8] V. Vajaitu, Approximation theorems and homology of q -Runge domains, J. reine angew. Math. 449 (1994), 179-199.